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THE SOLUTION OF EVOLUTIONARY GAMES USING THE THEORY OF HAMILTON–JACOBI EQUATIONS†

A. M. TARAS'YEV

Ekaterinburg

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A dynamical model of a non-antagonistic evolutionary game for two coalitions is considered. The model features an infinite time span and discounted payoff functionals. A solution is presented using differential game theory. The solution is based on the construction of a value function for auxiliary antagonistic differential games and uses an approximate grid scheme from the theory of generalized solutions of the Hamilton–Jacobi equations. Together with the value functions the optimal guaranteeing procedures for control on the grid are computed and the Nash dynamic equilibrium is constructed. The behaviour of trajectories generated by the guaranteeing controls is investigated. Examples are given.

1. STATEMENT OF THE PROBLEM AND METHODS OF SOLVING IT

We consider a game-theory model of the interaction between two large groups (coalitions or populations) of participants (individuals) over a prolonged (infinite) time interval. The dynamical system is motivated by differential game models [1–4] and evolutionary games [5–12] associated with problems of economic change [6–9] and population dynamics [10]. We use approaches [12] for constructing controllable dynamics and formulations of non-antagonistic differential games. The problem is formulated in terms of the theory of positional differential games [2–4] and is solved by the method of generalized (minimax, viscous) solutions of the Hamilton–Jacobi equations [13–16].

It is assumed that the participants in each coalition have just two actions (strategies) and that at any instant of the time they can only adhere to one of them. It is assumed that the participants of the different coalitions meet (form a game pair) in a random manner. Here the payoffs of the coalition participants are determined by payoff matrices. Local (short-term) payoffs of the coalitions are specified by the corresponding mathematical expectations (mean payoffs). The global (long-term) interests of the coalitions are represented by integrals of the mathematical expectations over an infinite time interval with an appropriate discounting coefficient [6, 17, 18]. The participants can change their actions in accordance with information signals. The corresponding evolutionary dynamics of the coalitions are described by a system of differential equations with controlling parameters (signals). The controls can be chosen in any manner according to feedback based on information on the unfolding dynamical position of the system. The aims of the coalitions are to maximize their own global interests. The problem of the corresponding non-antagonistic game is to construct optimal guaranteeing positional control procedures for the coalitions and the Nash dynamical equilibrium situation.

The non-antagonistic game is considered in terms of the theory of positional differential games [2–4]. Following [4] the Nash dynamical equilibrium situation is constructed using solutions of two auxiliary antagonistic differential games. The solution of the antagonistic games is related to the construction of value functions which are generalized solutions to first-order partial differential equations [13–16, 19]. The value functions are calculated approximately in terms of the theory of generalized solutions of the Hamilton–Jacobi equations. The appropriate computational procedure is an approximational grid scheme with suitable finite-difference operators [17–18, 20–27]. The values of the optimal guaranteeing synthesis are calculated in parallel with the value functions. It is important to note that the optimal guaranteeing control procedures for the coalitions generate system trajectories that converge either to a Nash static equilibrium situation or to positions in which the values of the global payoffs are better than at points of static equilibrium.

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2. THE GAME DYNAMICS MODEL

We consider a dynamical system describing the game interactions of two coalitions of participants. Using the well-known economic interpretation [5] one can suppose, for example, that one coalition consists of sellers, and the other of purchasers. At each instant of time the participants can choose one of two actions: the sellers can try to sell at a high or a low price, and the buyers can buy or refuse to buy. The action of the sellers is denoted by the index i: the value $i = 1$ corresponds to a high price, and $i = 2$ to a low price. The action of the buyers is similarly denoted by j: the value $j = 1$ corresponds to buying and $j = 2$ to refusing to buy.

We consider an arbitrary pair consisting of participants from the two coalitions. This pair can be represented as a situation (i, j) generated by actions i and j. We take the payoffs of the participants of the first and second coalitions to be determined by the coefficients a_{ij}^m of the matrices $A^m = \{a_{ij}^m\}$ $(m = 1, 2)$, respectively.

Let the first coalition consist of N participants. The symbol $N_i(t)$ denotes the number of participants choosing action i, $i = 1, 2$ at time t. Obviously $N = N_1(t) + N_2(t)$. The second coalition similarly consists of M participants and $M_j(t)$ of them adopt strategy j, $j = 1, 2$ so that $M = M_1(t) + M_2(t)$.

We suppose there is a multistep dynamical process in which the participants can change their actions, described by the system of equations

$$N_i(t+h) = N_i(t) + (-1)^i (n_{12}(t)h - n_{21}(t)h), \quad i = 1,2 \tag{2.1}$$

$$M_j(t+h) = M_j(t) + (-1)^j (m_{12}(t)h - m_{21}(t)h), \quad j = 1,2$$

A feature of the dynamics of (2.1) is that the number of participants which can change their action at time t is proportional to the step h, $0 < h \leqslant 1$. More precisely, the numbers $n_{kl}(t)h$ and $m_{kl}(t)h$ denote the number of participants in each coalition which change from action k to action l, $k, l = 1, 2$, $k \neq l$.

This fact, that only a fraction of members of the coalitions, proportional to the step h, can change their action at the actual time t, has the following interpretation. For example, such an inertia can be explained by only a small number of the individuals being active and amenable to changing their behaviour. An alternative explanation would be the presence of a "queue" in those cases when large groups of participants wish to change their actions.

We also introduce the following natural restrictions on the numbers of participants $n_{kl}(t)$, $m_{kl}(t)$ potentially wishing to change their strategies

$$0 \leqslant n_{kl}(t) \leqslant N_k(t), \quad 0 \leqslant m_{kl}(t) \leqslant M_k(t) \tag{2.2}$$

We assume that at each time t the members of the different coalitions form game pairs in a random manner with equal probabilities. The probability that a randomly chosen pair forms the situation (i, j) is equal to

$$p_{ij}(t) = \frac{N_i(t) M_j(t)}{NM} \tag{2.3}$$

It is easy to check the probabilistic properties

$$p_{ij}(t) \geqslant 0, \quad \sum_{i,j} p_{ij}(t) = 1, \quad i, j = 1,2 \tag{2.4}$$

One can transfer from the multistep system (2.1) linking the quantities $N_i(t + h)$, $M_j(t + h)$ with the $N_i(t)$, $M_j(t)$ to a system for the probabilities $p_{ij}(t + h)$ and $p_{ij}(t)$

$$p_{ij}(t+h) = \frac{N_i(t+h) M_j(t+h)}{NM} = p_{ij}(t) - p_{ij}(t) u_i(t) h + \tag{2.5}$$

$$+ p_{kj}(t) u_k(t) h - p_{ij}(t) \upsilon_j(t) h + p_{il}(t) \upsilon_l(t) h + \varphi(t) h^2$$

$$u_i(t) = \frac{n_{ik}(t)}{N_i(t)}, \quad \upsilon_j(t) = \frac{m_{jl}(t)}{M_j(t)}$$

$$\varphi(t) = \frac{(-n_{ik}(t) + n_{ki}(t))(m_{jl}(t) + m_{lj}(t))}{NM}, \quad |\varphi(t)| \leqslant 1$$

$$i \neq k, \quad j \neq l, \quad i, j, k, l = 1, 2$$

We transfer from the multistep system (2.5) to a system of ordinary differential equations

$$\begin{aligned} \dot{x}_1 &= -x_1 u_1 + x_3 u_2 - x_1 \upsilon_1 + x_2 \upsilon_2 \\ \dot{x}_2 &= -x_2 u_1 + x_4 u_2 + x_1 \upsilon_1 - x_2 \upsilon_2 \\ \dot{x}_3 &= x_1 u_1 - x_3 u_2 - x_3 \upsilon_1 + x_4 \upsilon_2 \\ \dot{x}_4 &= x_2 u_1 - x_4 u_2 + x_3 \upsilon_1 - x_4 \upsilon_2 \end{aligned} \tag{2.6}$$

$$(x_1 = p_{11}, \quad x_2 = p_{12}, \quad x_3 = p_{21}, \quad x_4 = p_{22})$$

$$0 \leqslant u_i \leqslant 1, \quad i = 1, 2; \quad 0 \leqslant \upsilon_j \leqslant 1, \quad j = 1, 2 \tag{2.7}$$

Remarks. 1. The quantities u_i, υ_j in (2.6) are dimensionless, and the 0 and 1 in (2.7) may be interpreted as controlling signals.

2. The dynamical system (2.6), (2.7) has two first integrals

$$x_1 + x_2 + x_3 + x_4 = 1, \quad x_1 x_4 - x_2 x_3 = 0 \tag{2.8}$$

and satisfies the integral constraints

$$0 \leqslant x_i \leqslant 1, \quad i = 1, 2, 3, 4 \tag{2.9}$$

Hence it reduces to the second-order system

$$\dot{x} = -x u_1 + (1 - x) u_2, \quad \dot{y} = -y \upsilon_1 + (1 - y) \upsilon_2 \tag{2.10}$$

Here $x = x_1 + x_2$ ($y = x_1 + x_3$) is the probability that a player from the first (second) coalition has chosen the first strategy.

3. The controlling parameters u, υ

$$u = x(1 - u_1) + (1 - x) u_2, \quad 0 \leqslant u_i \leqslant 1, \quad i = 1, 2$$

$$\upsilon = y(1 - \upsilon_1) + (1 - y) \upsilon_2, \quad 0 \leqslant \upsilon_j \leqslant 1, \quad j = 1, 2$$

satisfy the constraints

$$u \in I, \quad \upsilon \in I, \quad I = [0, 1] \tag{2.11}$$

and system (2.10) is equivalent to the system

$$\dot{x} = -x + u, \quad \dot{y} = -y + \upsilon \tag{2.12}$$

System (2.12) with conditions (2.11) satisfies the integral restrictions

$$x \in I, \quad y \in I \tag{2.13}$$

The maximum system velocities V_1 and V_2 for each coordinate under the restrictions (2.11) and (2.13) are unity

$$V_1 = \max_{x,u} |-x + u| = 1, \quad V_2 = \max_{y,\upsilon} |-y + \upsilon| = 1 \tag{2.14}$$

4. System (2.12) possesses the asymptotic stability property

$$|x_1(t) - x_2(t)| \leqslant |x_1 - x_2| e^{-t}, \quad |y_1(t) - y_2(t)| \leqslant |y_1 - y_2| e^{-t} \tag{2.15}$$

Here $(x_i(t), y_i(t))$ is a trajectory of system (2.12) generated by a pair of measurable controls $(u(s), \upsilon(s))$, $u(s)$: $[0, +\infty) \to I$, $\upsilon(s)$: $[0, +\infty) \to I$ from the initial position $(x_i, y_i) \in I \times I$, $i = 1, 2$

$$x_i(t) = x_i e^{-t} + \int_0^t e^{(s-t)} u(s) ds, \quad y_i(t) = y_i e^{-t} + \int_0^t e^{(s-t)} \upsilon(s) ds$$

3. LOCAL AND GLOBAL PAYOFF FUNCTIONALS

We consider the problem of evaluating the interests of the coalitions. It is natural to suppose that the local (short-term) payoff is given by the mathematical expectation associated with the corresponding payoff matrix and is calculated at the actual time t. Specifically, the quality of state $(x(t), y(t))$ of dynamical system (2.12) is valued for the first ($m = 1$) and second ($m = 2$) coalitions by the mathematical expectations

$$E_m(x(t), y(t)) = a_{11}^m x(t)y(t) + a_{12}^m x(t)(1-y(t)) + a_{21}^m (1-x(t))y(t) + \tag{3.1}$$
$$+a_{22}^m (1-x(t))(1-y(t)) = C_m x(t)y(t) - \alpha_1^m x(t) - \alpha_2^m y(t) + a_{22}^m$$

Here

$$C_m = a_{11}^m - a_{12}^m - a_{21}^m + a_{22}^m, \quad \alpha_1^m = a_{22}^m - a_{12}^m, \quad \alpha_2^m = a_{22}^m - a_{21}^m \tag{3.2}$$

We recall that the quantities (3.2) govern the coordinates of the Nash equilibrium situation in a bimatrix game [5].

We consider the dynamical system (2.12) in an infinite time interval $[0, +\infty)$, as is assumed in the theory of evolutionary games (evolutionary changes) [6–12]. Let $(x(\cdot), y(\cdot)) = \{(x(t), y(t)): t \in [0, +\infty)]$ be an arbitrary trajectory of system (2.12). The value of the trajectory is estimated by improper integral functionals with discounting. The functionals have the form

$$J_m = J_m(x(\cdot), y(\cdot)) = \int_0^{+\infty} e^{-\lambda t} E_m(x(t), y(t))dt, \quad m = 1, 2 \tag{3.3}$$

It is natural to suppose that the functionals (3.3) specify a long-term (global) payoff for the coalition. The parameter $\lambda > 0$ is called the discounting coefficient and ensures the discounting of short-term payoffs at future times. Functionals of this sort are traditional in models of mathematical economics [6, 17]. We note that integrals (3.3) coverage because the functions E_m are bounded.

Integrals (3.3) can be normalized by multiplying them by λ and interpreting the result as a special kind of average of the mathematical expectation E_m^* over the interval $[0, +\infty)$

$$E_m^* = \lambda J_m = \sum_{i,j} a_{ij}^m p_{ij}^*, \quad p_{ij}^* = \lambda \int_0^{+\infty} e^{-\lambda t} p_{ij}(t)dt \tag{3.4}$$
$$0 \leqslant p_{ij}^* \leqslant 1, \quad \sum_{i,j} p_{ij}^* = 1, \quad i, j = 1, 2$$

Here p_{ij}^* is the probability of the game situation (i, j) averaged over the infinite time interval $[0, +\infty)$.

4. NASH DYNAMICAL EQUILIBRIUM

The aims of the coalitions are to maximize their functionals (3.3). We shall consider a formulation of the associated non-coalition game using the theory of positional differential games [2–4]. According to one of the constructions in [4] the equilibrium for a non-coalition differential game (2.12), (3.3) can be obtained in the class of control procedures $U = u(t, x, y, \varepsilon)$, $V = \upsilon(t, x, y, \varepsilon)$ by the feedback principle in the solution of guaranteed optimal control problems for each coalition.

We will give the formulation for defining the Nash equilibrium situation.

Definition 1. The pair of positional strategies (U^0, V^0) is said to be a Nash equilibrium (ε-equilibrium) for the given initial position $(x_0, y_0) \in I \times I$ if for any positional strategies U and V and for any trajectories generated by the pairs of strategies (x_0, y_0) from the point (U^0, V^0), (U, V^0), (U^0, V)

$$(x^0(\cdot), y^0(\cdot)) \in X_0, \quad (x_m(\cdot), y_m(\cdot)) \in X_m$$
$$X_0 = X(x_0, y_0, U^0, V^0), \quad X_1 = X(x_0, y_0, U, V^0), \quad X_2 = X(x_0, y_0, U^0, V)$$

the inequalities

$$J_m(x^0(\cdot), y^0(\cdot)) \geqslant J_m(x_m(\cdot), y_m(\cdot)) - \varepsilon \tag{4.1}$$

are satisfied (with the parameter $\varepsilon = 0$ for equilibrium and $\varepsilon > 0$ for εequilibrium).

We will indicate a possible construction in which the Nash equilibrium as specified in Definition 1 is composed of solutions to auxiliary problems of optimal guaranteed control (antagonistic games). We consider two differential games Γ_m. In game Γ_m the problem for coalition m is to maximize the functional J_m, $m = 1, 2$ (3.3) along the trajectories of system (2.12), and the aim of the second coalition is the opposite.

We suppose that the games Γ_m are solved, i.e. the value functions $w_m = w_m(x, y)$, $(x, y) \in I \times I$ are calculated, the optimal guaranteeing positional control procedures ("guaranteeing strategies") $U_1^0 = u_1^0(t, x, y, \varepsilon)$, $V_2^0 = \upsilon_2^0(t, x, y, \varepsilon)$ maximizing J_m are constructed, together with the adverse actions of the coalitions towards one another—the positional control procedures ("punishing strategies") $U_2^0 = u_2^0(t, x, y, \varepsilon)$, $V_1^0 = \upsilon_1^0(t, x, y, \varepsilon)$ minimizing J_m.

Suppose that an arbitrary initial position $(x_0, y_0) \in I \times I$ has been specified, the parameter $\varepsilon > 0$, and the trajectory $(x^0(\cdot), y^0(\cdot)) \in X(x_0, y_0, U_1^0, V_2^0)$, generated by the guaranteeing strategies U_1^0, V_2^0 has been determined. It is natural to say that the trajectory $(x^0(\cdot), y^0(\cdot))$ is acceptable to both coalitions. We specify a time $T_\varepsilon > 0$ by the conditions

$$T_\varepsilon > \frac{1}{\lambda} \ln\left(\frac{R}{\lambda\varepsilon}\right), \quad R = \max_{(x,y)} \max_m |E_m(x, y)|$$

In the interval $[0, T_\varepsilon]$ we define an acceptable step-by-step motion $(x^\varepsilon(\cdot), y^\varepsilon(\cdot))$ satisfying the condition

$$\max_{t \in [0, T_\varepsilon]} \|(x^0(t), y^0(t)) - (x^\varepsilon(t), y^\varepsilon(t))\| < \varepsilon$$

The symbols $u_1^\varepsilon(t)$: $[0, T_\varepsilon) \to I$ and $\upsilon_2^\varepsilon(t)$: $[0, T_\varepsilon) \to I$ denote realizations of guaranteeing strategies U_1^0 and V_2^0 for the motion $(x^\varepsilon(\cdot), y^\varepsilon(\cdot))$.

We construct position strategies U^0, V^0 from the guaranteeing strategies U_1^0 and V_2^0 and the punishing strategies U_2^0 and V_1^0

$$U^0 = u^0(t, x, y, \varepsilon) = \begin{cases} u_1^\varepsilon(t), & \text{if} \quad \|(x, y) - (x^\varepsilon(t), y^\varepsilon(t))\| < \varepsilon \\ u_2^0(t, x, y, \varepsilon), & \text{otherwise} \end{cases} \tag{4.2}$$

$$V^0 = \upsilon^0(t, x, y, \varepsilon) = \begin{cases} \upsilon_2^\varepsilon(t), & \text{if} \quad \|(x, y) - (x^\varepsilon(t), y^\varepsilon(t))\| < \varepsilon \\ \upsilon_1^0(t, x, y, \varepsilon), & \text{otherwise} \end{cases} \tag{4.3}$$

As in [4] one can prove the following assertion.

Proposition 1. The pair of positional strategies (U^0, V^0) specified by relations (4.2), (4.3) is a Nash ε-equilibrium in the sense of Definition 1.

Remark 5. In the Nash equilibrium pair of positional strategies (4.2), (4.3) we used the acceptable trajectory $(x^\varepsilon(\cdot)), y^\varepsilon(\cdot))$ generated by the guaranteeing strategies U_1^0, V_2^0. According to [4], when constructing (4.2) and (4.3) one can use other trajectories, the manifolds of which exhaust all the possible Nash equilibrium situations.

Remark 6. Depending on the formulation of the problem, the parameter ε can be interpreted either as the level of risk or the degree of confidence (altruism) of the coalition.

5. VALUE FUNCTIONS FOR GAMES OF UNRESTRICTED LENGTH

It follows from relations (4.2) and (4.3) that the fundamental constructions for obtaining the equilibrium strategies U^0, V^0 are solutions of single-type differential games Γ_m. To fix our ideas we will consider a differential game Γ_1 of this sort. In this game one must guarantee the maximization of functional J_1 (3.3) along the trajectories $(x(\cdot), y(\cdot))$ of system (2.12). It is known [2, 3] that a value function $(x_0, y_0) \to w_1(x_0, y_0)$ exists in the differential game (2.12), (3.3) which associates every position (x_0, y_0) with the value of the saddle point

$$w_1(x_0, y_0) = \sup_U \inf_{(x(\cdot), y(\cdot)) \in X(x_0, y_0, U)} J_1(x(\cdot),\ y(\cdot)) = \\ = \inf_V \sup_{(x(\cdot), y(\cdot)\ V) \in X(x_0, y_0, V)} J_1(x(\cdot),\ y(\cdot)) \tag{5.1}$$

The symbols $X(x_0, y_0, U)$, $X(x_0, y_0, V)$ denote trajectories of system (2.12) generated by positional strategies $U = u(t, x, y, \varepsilon)$, $V = v(t, x, y, \varepsilon)$.

According to [2, 3] the guaranteeing strategy U_1^0 and the punishing strategy V_1^0 are determined by the value function w_1.

We will indicate some properties of the value function w_1.

Property 1. The value function w_1 is defined and bounded in the unit square $I \times I$

$$\max_{(x,y) \in I \times I} |w_1(x,y)| \leq \frac{K}{\lambda}, \quad K = \max_{(x,y) \in I \times I} |E_1(x,y)| \tag{5.2}$$

In a differential game of unbounded length of general form one can establish [17, 18] a Hölder continuity property for the value function. In the case of dynamical system (2.12) possessing the asymptotic stability property (Remark 4) it can be proved that the function w_1 satisfies a Lipschitz condition. In particular, we have the following assertion.

Property 2. The Lipschitz condition

$$|w_1(x_1, y_1) - w_1(x_2, y_2)| \leq \frac{K}{1+\lambda}(|x_1 - x_2| + |y_1 - y_2|) \tag{5.3}$$

is satisfied by the value function w_1.

6. THE HAMILTON–JACOBI EQUATION AND DIFFERENTIAL INEQUALITIES FOR THE VALUE FUNCTION

The fundamental properties of the value function w_1 are stability properties expressing the optimality principle of dynamical programming. We recall that stability properties amount to the existence of non-decreasing and non-increasing directions for the value function coordinated with the directions of the velocities of the dynamical system. At differentiable points of the value function the optimality principle reduces to a Hamilton–Jacobi (Bellman–Isaacs) type first-order partial differential equation. In the differential game (2.12), (3.3) the Hamilton–Jacobi equation has the form

$$-\lambda w_1(x,y) - \frac{\partial w_1}{\partial x} x - \frac{\partial w_1}{\partial y} y + \max\left\{0, \frac{\partial w_1}{\partial x}\right\} + \min\left\{0, \frac{\partial w_1}{\partial y}\right\} + E_1(x,y) = 0 \tag{6.1}$$

The expression

$$H_1(x,y,s) = -s_1 x - s_2 y + \max\{0, s_1\} + \min\{0, s_2\} + E_1(x,y) \tag{6.2}$$

is called the Hamiltonian of problem (2.12), (3.3), $s = (s_1, s_2)$.

At points where the value function is non-differentiable, the following inequalities must be satisfied by derivatives along the directions (dual derivatives) [14, 18, 19]

$$D^* w_1(x,y)|(s) = \sup_{h \in R^2} (\langle s, h\rangle - \partial_- w_1(x,y)|(h)) \geq -\lambda w_1(x,y) + H_1(x,y,s) \tag{6.3}$$

$$D_* w_1(x,y)|(s) = \inf_{h \in R^2} (\langle s, h\rangle - \partial_+ w_1(x,y)|(h)) \leq -\lambda w_1(x,y) + H_1(x,y,s) \tag{6.4}$$

The lower and upper directional derivatives for the function w_1 are governed by the relations

$$\partial_- w_1(x,y)|(h) = \liminf_{\delta \downarrow 0} \frac{w_1(x + \delta h_1, y + \delta h_2) - w_1(x,y)}{\delta}$$

$$\partial_+ w_1(x,y)|(h) = \limsup_{\delta \downarrow 0} \frac{w_1(x+\delta h_1, y+\delta h_2) - w_1(x,y)}{\delta}$$

Remark 7. We know [14, 17–19] that inequalities (6.3) and (6.4) uniquely define the generalized solution of Eq. (6.1) (the value function) in the class of bounded functions satisfying the Lipschitz condition. Inequality (6.4) expresses the u-stability property of the function w_1, and (6.3) the υ-stability property.

Remark 8. For piecewise-smooth functions

$$w_1(x,y) = \min_i \max_j \varphi_{ij}(x,y) = \max_j \min_i \varphi_{ij}(x,y)$$

$$w_1(x_*,y_*) = \varphi_{ij}(x_*,y_*), \quad i \in I(x_*,y_*), \quad j \in J(x_*,y_*), \quad (x,y) \in O_\varepsilon(x_*,y_*)$$

$$O_\varepsilon(x_*,y_*) = \{(x,y) \in I \times I : \max\{|x-x_*|, |y-y_*|\} < \varepsilon\}$$

the directional and dual derivatives can be computed from the formulae

$$\partial_- w_1(x,y)|(h) = \partial_+ w_1(x,y)|(h) = \min_i \max_j \langle b_{ij}, h\rangle = \max_j \min_i \langle b_{ij}, h\rangle$$

$$b_{ij} = \left(\frac{\partial \varphi_{ij}}{\partial x}, \frac{\partial \varphi_{ij}}{\partial y}\right), \quad h = (h_1, h_2)$$

$$D^* w_1(x_*,y_*)|(s) = \begin{cases} 0, & \text{if } s \in C \\ +\infty, & \text{if } s \notin C \end{cases}, \quad D_* w_1(x_*,y_*)|(s) = \begin{cases} 0, & \text{if } s \in D \\ -\infty, & \text{if } s \notin D \end{cases}$$

$$C = \bigcap_i B_i, \quad B_i = \mathrm{co}_j\{b_{ij}\}, \quad D = \bigcap_j B_j, \quad B_j = \mathrm{co}_i\{b_{ij}\}$$

We will indicate cases when the value function w_1 is differentiable and can be found by the method of undetermined coefficients.

Proposition 2. We assume that the coefficients of the matrix $A^1 = \{a^1_{ij}\}$ satisfy the inequalities

$$\frac{\alpha^1_1}{C_1} \geqslant \frac{1+\lambda}{2+\lambda}, \quad \alpha^1_2 \leqslant 0, \quad C_1 > 0 \tag{6.5}$$

Then the value function w_1 is given the relations

$$w_1(x,y) = Gxy - \gamma_1 x - \gamma_2 y + g \tag{6.6}$$

$$G = \frac{C_1}{2+\lambda}, \quad \gamma_1 = \frac{\alpha^1_1}{1+\lambda}, \quad \gamma_2 = \frac{\alpha^1_2}{1+\lambda}, \quad g = \frac{a^1_{22}}{\lambda}$$

Here the extrema is Eq. (6.1) are zero

$$\max\left\{0, \frac{\partial w_1}{\partial x}\right\} = 0, \quad \min\left\{0, \frac{\partial w_1}{\partial y}\right\} = 0$$

which shows the preservation of the sign of the gradient components $\nabla w_1 = (\partial w_1/\partial x, \partial w_1/\partial y)$ of the functions w_1 in the square $I \times I$

$$\frac{\partial w_1}{\partial x} = Gx - \gamma_1 \leqslant 0, \quad \frac{\partial w_1}{\partial y} = Gx - \gamma_2 \geqslant 0 \tag{6.7}$$

Remark 9. Examples of matrices satisfying conditions (6.5) include not only matrices with a dominant second row, but also matrices with a dominant first column. For example, the matrix

$$A = \begin{Vmatrix} 3 & 0 \\ 2 & 2 \end{Vmatrix}$$

satisfies inequalities (6.5) when $0 \leq \lambda \leq 1$.

Conditions similar to relations (6.5) may be obtained for other combinations of inequalities in (6.7). In the case when, for example, the conditions

$$0<\frac{\alpha_1^1}{C_1}<1, \quad 0<\frac{\alpha_2^1}{C_1}<1, \quad C_1 \neq 0$$

are satisfied, the value function w_1 is not differentiable and has no analytic description. In this case its structure is rather complex. Below we therefore present an approximation scheme for constructing the value function w_1 as a generalized solution of the Hamilton–Jacobi equation (6.1).

7. APPROXIMATION SCHEMES FOR CONSTRUCTING VALUE FUNCTIONS AND OPTIMAL POSITIONAL STRATEGIES

We consider a discrete approximation to Eq. (6.1). We fix the step $h \in (0, 1/\lambda)$ for subdividing the time interval $[0, +\infty)$.

Definition 2. An equation of the form

$$-\underline{w}_{1,h}(x,y)+hE_1(x,y)+(1-\lambda h)\max_{u\in I}\min_{\upsilon\in I}\underline{w}_{1,h}(x+h(-x+u),y+h(-y+\upsilon))=0 \tag{7.1}$$

for the function $\underline{w}_{1,h}$: $I\times I \to R$ is said to be a discrete approximation of the Hamilton–Jacobi (7.1), and its solution $\underline{w}_{1,h}$ is called a lower approximation of the generalized solution (the value function) w_1 of Eq. (7.1).

We derive the following assertion about the convergence of the approximate solution $\underline{w}_{1,h}$ [17, 18]. We also note that similar approximation problems for value functions were considered in [26, 27].

Proposition 3. A unique solution $\underline{w}_{1,h}$ of Eq. (7.1) exists. As $h\downarrow 0$ the functions $\underline{w}_{1,h}$ converge to the value function w_1 in the metric of the space of continuous functions and the estimate

$$\max_{(x,y)\in I\times I}|\underline{w}_{1,h}(x,y)-w_1(x,y)|< Lh^{1/2} \tag{7.2}$$

holds. The solution $\underline{w}_{1,h}$ can be found by the method of successive approximations

$$\underline{w}_{1,h}^{n}(x,y)=(\Pi_*\underline{w}_{1,h}^{n-1})(x,y) \tag{7.3}$$

$$(\Pi_* w)(x,y)=hE_1(x,y)+(1-\lambda h)\max_{u\in I}\min_{\upsilon\in I} w(x+h(-x+u),y+h(-y+\upsilon)) \tag{7.4}$$

As an initial approximation for the iterative procedure (7.3), (7.4) one can choose any bounded and Lipschitz-continuous function. For example, it is natural to put

$$\underline{w}_{1,h}^{0}(x,y)=0, \quad \text{or} \quad \underline{w}_{1,h}^{0}(x,y)=E_1(x,y) \tag{7.5}$$

Π_* is a contraction operator with contraction coefficient $(1-\lambda h)$, and hence the estimate

$$\max_{(x,y)\in I\times I}|\underline{w}_{1,h}(x,y)-\underline{w}_{1,h}^{n}(x,y)|\leq \frac{K}{\lambda}(1-\lambda h)^n \tag{7.6}$$

holds. For a sufficiently large number of iterations m estimates (7.2) and (7.6) ensure that the condition

$$\max_{(x,y)\in I\times I}|\underline{w}_{1,h}^{m}(x,y)-w_1(x,y)|< Lh^{1/2} \tag{7.7}$$

is satisfied.

We note that in procedure (7.3), (7.4) the control $u_{1,h}^m$ producing the exterior maximum

$$u_{1,h}^m=u_{1,h}^m(x,y)=\arg\max_{u\in I}\min_{\upsilon\in I}\underline{w}_{1,h}^{m-1}(x+h(-x+u),y+h(-y+\upsilon)) \tag{7.8}$$

in (7.4) is calculated along with the approximate values $\underline{w}^m_{1,h}(x,y)$ for the value function $w_1(x,y)$.

It can be shown that the positional strategy $u^m_{1,h}$ (7.8) is an approximation to the guaranteeing strategy U^0_1 for the first coalition. The following assertion is moreover valid.

Proposition 4. For any $\varepsilon > 0$ one can find a subdivision step $h \in (0, 1/\lambda)$ and an iteration number m such that for all trajectories $(x(\cdot), y(\cdot))$, generated by the positional strategies $u^m_{1,h}$ and arbitrary measurable controls $\upsilon(s)$: $[0, +\infty) \to I$ from the initial condition (x_0, y_0), the limit

$$\int_0^{mh} e^{-\lambda t} E_1(x(t), y(t))dt > w_1(x_0, y_0) - \varepsilon \tag{7.9}$$

holds.

Alongside Eq. (7.1), which contains a maximum operator, one can consider an equation with a minimax operator for the upper approximation $\overline{w}_{1,h}$. The solution $\overline{w}_{1,h}$ can also be found by the method of successive approximations from formulae for the approximation $\overline{w}^m_{1,h}$ similar to (7.3)–(7.5). Here the punishment strategy υ^0_1 of the second coalition approximates the positional strategy $\upsilon^m_{1,h}$ which achieves outer minimum and minimax relations similar to (7.4) and has the form

$$\upsilon^m_{1,h} = \upsilon^m_{1,h}(x, y) = \arg\min_{\upsilon \in I} \max_{u \in I} \overline{w}^{m-1}_{1,h}(x + h(-x + u), y + h(-y + \upsilon)) \tag{7.10}$$

Remark 10. For the upper approximation $\overline{w}_{1,h}$ and punishment strategy $\upsilon^m_{1,h}$ one can formulate assertions similar to Propositions 3 and 4.

Remark 11. In the second game Γ_2 the approximations $\underline{w}^m_{2,h}$, $\overline{w}^m_{2,h}$ for the value function w_2 and the approximations $\upsilon^m_{2,h}$, $u^m_{2,h}$ for the guaranteeing strategy V^0_2 and punishment strategy U^0_2 are calculated similarly.

8. GRID IMPLEMENTATIONS OF ALGORITHMS FOR CONSTRUCTING VALUE FUNCTIONS AND OPTIMAL POSITIONAL STRATEGIES

To implement the iterational procedure (7.3)–(7.5) and similar procedures for the equation with the minimax operator numerically, a grid approximation is used for the iterational functions $\underline{w}^n_{1,h}$, $\underline{w}^n_{2,h}$, $\overline{w}^n_{1,h}$, $\overline{w}^n_{1,h}$ and associated positional strategies $u^m_{1,h}$, $\upsilon^m_{2,h}$, $\upsilon^m_{1,h}$, $u^m_{2,h}$. The grid approximations for these functions are denoted by the same symbols. We note that formally the calculations for the maximin and minimax should be performed at all points (x, y) of the phase state square $I \times I$. In order to make the procedure finite a grid is introduced and the calculations are only performed at its nodes, and are linearly interpolated over the whole square in accordance with its specified triangulation Ω.

We will describe the proposed numerical procedure. Let the following discretization steps be given: the time subdivision interval h, the subdivision steps $\Delta x(\Delta y)$ in the square $I \times I$ for variables x (y), and the subdivision steps $\Delta p(\Delta q)$ in the interval I for the control u (υ).

We shall assume that the steps h, Δx, Δy, Δp, Δq are related as follows:

$$\Delta x = \psi_1(h), \quad \Delta y = \psi_2(h), \quad \Delta p = \psi_3(h), \quad \Delta q = \psi_4(h) \tag{8.1}$$

Here $\psi_i(h)$ are infinitesimally small functions: $\psi_i(h)\downarrow 0$ when $h \downarrow 0$, $i = 1, 2, 3, 4$.

We consider the fixed grid GR

$$GR = \{(x_i, y_j): \ x_i = i\Delta x, \ \ y_j = j\Delta y\}, \ \ (x_i, y_j) \in I \times I$$

All the calculations are performed at its nodes (x_i, y_j). For example, the lower approximation in Eq. (7.1) is calculated from the formulae

$$\underline{w}^n_{1,h}(x_i, y_j) = hE_1(x_i, y_j) + (1 - \lambda h)\max_k \min_l \underline{w}^{n-1}_{1,h}(x_{ik}, y_{jl}) \tag{8.2}$$
$$x_{ik} = x_i + h(-x_i + k\Delta p), \quad y_{jl} = y_j + h(-y_j + l\Delta q)$$

Here the values of the functions at the points (x_{ik}, y_{jl}) are linear interpolations of the values given at the nodes (x_i, y_j) of the grid GR according to the specified triangulation Ω of the square $I \times I$ into simplexes of type S_+ and S_-

$$S_+ = S_+(x_i, y_j) = \text{co}\{(x_i, y_j),\ (x_i + \Delta x, y_j),\ (x_i, y_j + \Delta y)\}$$

$$S_- = S_-(x_i, y_j) = \text{co}\{(x_i + \Delta x, y_j),\ (x_i, y_j + \Delta y),\ (x_i + \Delta x, y_j + \Delta y)\}$$

The grid functions $\underline{w}^n_{2,h}(x_i, y_j)$, $w^n_{2,h}(x_i, y_j)$, $\underline{w}^n_{1,h}(x_i, y_j)$ are similarly from the corresponding maximum and minimax formulae.

Using the results of [20–23] one can prove the following assertion on the convergence of grid approximation schemes of the form (8.2)

Proposition 5. Suppose the functions in (8.1) are given by the relations

$$\psi_i(h) = K_i h^{1+a}, \quad a \geqslant 0, \quad i = 1,2, \quad \psi_j(h) = K_j h, \quad j = 3,4 \tag{8.3}$$

i.e. the functions $\psi_1(h)$, $\psi_2(h)$ are either linear in the variable h (when $a = 0$) or have a higher order of smallness (when $a > 0$), while the functions $\psi_3(h)$, $\psi_4(h)$ are linear. Then grid approximation scheme (8.2) converges when $n \to \infty$, $h \downarrow 0$ and the convergence estimate is a quantity of order $h^{1/2}$.

We will consider the possibility of using the parameters k and l which realize the outer extrema in the maximin and minimax relations of the form (8.2) as approximations for the guaranteeing strategies U^0_1, V^0_2 and punishment strategies U^0_2, V^0_1. We give a relation for the grid function $u^m_{1,h}$ approximating the strategy U^0_1

$$u^m_{1,h}(x_i, y_j) = k^*(x_i, y_j)\Delta p \tag{8.4}$$

Here $k^*(x_i, y_j)$ is the argument of the outer extremum in (8.2).

The approximate strategies $\upsilon^m_{2,h}$, $u^m_{2,h}$, $\upsilon^m_{1,h}$, are determined in the same way as (8.4) from the corresponding maximin and minimax formulae.

We note that the strategies $u^m_{1,h}$, $\upsilon^m_{2,h}$, $u^m_{2,h}$, $\upsilon^m_{1,h}$, are defined only at the nodes (x_i, y_j) of the grid GR.

We will consider two methods of interpolating these values: piecewise-constant and linear. One can prove the following assertion regarding, for example, the strategy $u^m_{1,h}$.

Proposition 6. Let $a > 0$ in (8.3), i.e. $\psi_1(h)$, $\psi_2(h)$ are functions of higher order of smallness than h. Then for any $\varepsilon > 0$ one can find a subdivision step $h \in (0, 1/\lambda)$ and an iteration number m such that for all trajectories $(x(\cdot), y(\cdot))$ generated by piecewise-constant interpolations u^{pc}_1 of the strategy values $u^m_{1,h}$ (x_i, y_j) (8.4) and an arbitrary measurable control $\upsilon(s)$, $\upsilon \in I$ from the initial position (x_0, y_0), the estimate (7.9) is satisfied.

In order to formulate a similar assertion when $a = 0$ it is necessary to introduce the following additional condition.

Hypothesis 1 (*on the structure of the solution*). Let $a = 0$, i.e. $\psi_1(h) = K_1 h$, $\psi_2(h) = K_2 h$, $r > \max\{K_1, K_2, V_1, V_2\}$, $V_1 = V_2 = 1$. We consider the set

$$Z = \{(x_i^*, y_j^*) \in GR\colon\ u^m_{1,h}(\xi_1, \eta_1) \neq u^m_{1,h}(\xi_2, \eta_2),\ (\xi_k, \eta_k) \in O(x_i^*, y_j^*, rh),\ \ k = 1,2\}$$

$$O(x_i^*, y_j^*, rh) = \{(\xi, \eta) \in I \times I\colon\ \max\{|\xi - x_i^*|, |\eta - y_j^*|\} \leqslant rh\}$$

of those nodes (x_i^*, y_j^*) of the grid GR, in the rh-neighbourhood $O(x_i^*, y_j^*, rh)$ of which the values of the control $u^m_{1,h}$ (x_i, y_j), take different values. We assume that the linear interpolation $\underline{w}^{m-1}_{1,h}(\xi, \eta)$ of the values $\underline{w}^{m-1}_{1,h}(x_i, y_j)$ is a concave function apart from an infinitesimally small quantity rh^{1+b} $(b > 0)$ in the rh-neighbourhood $O(x_i^*, y_j^*, rh)$ of any point $(x_i^*, y_j^*) \in Z$

$$\gamma \underline{w}^{m-1}_{1,h}(\xi_1, \eta_1) + (1-\gamma)\underline{w}^{m-1}_{1,h}(\xi_2, \eta_2) \geqslant \underline{w}^{m-1}_{1,h}(\gamma\xi_1 + (1-\gamma)\xi_2, \gamma\eta_1 + (1-\gamma)\eta_2) - rh^{1+b}$$
$$0 \leqslant \gamma \leqslant 1,\ \ (\xi_k, \eta_k) \in O(x_i^*, y_j^*, rh),\ \ k = 1,2$$

When Hypothesis 1 is satisfied we have the following assertion.

Proposition 7. Suppose that $a = 0$ and Hypothesis 1 is satisfied in (8.3). Then for any $\varepsilon > 0$ one can find a subdivision step $h \in (0, 1/\lambda)$ and an iteration number m such that for all trajectories $(x(\cdot),$

$y(\cdot)$) generated by a linear interpolation u_1^L of the strategy values $u_{1,h}^m(x_i, y_j)$ (8.4) and an arbitrary measurable control $\upsilon(s)$, $\upsilon \in I$ from the initial position (x_0, y_0), limit (7.9) is satisfied.

9. RESULTS OF NUMERICAL EXPERIMENTS

The algorithms for constructing the value functions w_1 and w_2, guaranteeing strategies U_1^0, V_2^0, punishment strategies U_2^0, V_1^0, acceptable trajectories $(x^\varepsilon(\cdot), y^\varepsilon(\cdot))$ and Nash equilibrium strategies U^0, V^0 were implemented on a computer.

Two basic combinations of payoff matrices A^1 and A^2 were used in the numerical experiments, generating three or one Nash equilibrium situations in the corresponding bimatrix game [5]. We recall that three equilibria occur in "almost single-type" coalition interests: the zig-zags of acceptable situations have different orientations (left and right) and intersect at three points (equilibrium situations). One Nash equilibrium situation takes place for "almost antagonistic" coalition interests: the zig-zags of acceptable situations have the same orientations (both right or both left) and intersect at a single point (equilibrium situation).

In the first case (three equilibria) we considered the following single-type payoff matrices A^1 and A^2

$$A^1 = B_1 = \begin{Vmatrix} 11 & 2 \\ 3 & 6 \end{Vmatrix}, \quad C_1 = a_{11}^1 - a_{12}^1 - a_{21}^1 + a_{22}^1 = 12, \quad \alpha_1^1 = a_{22}^1 - a_{12}^1 = 4, \quad \alpha_2^1 = a_{22}^1 - a_{21}^1 = 3$$

$$A^2 = B_2 = \begin{Vmatrix} 3 & 1 \\ 0 & 4 \end{Vmatrix}, \quad C_2 = a_{11}^2 - a_{12}^2 - a_{21}^2 + a_{22}^2 = 6, \quad \alpha_1^2 = a_{22}^2 - a_{12}^2 = 3, \quad \alpha_2^2 = a_{22}^2 - a_{21}^2 = 4$$

In the corresponding antagonistic matrix games the saddle-point situations are the points $SP_1 = (1/4, 1/3)$ for matrix A^1 and $SP_2 = (2/3, 1/2)$ for matrix A^2. In the non-antagonistic bimatrix game there are three Nash equilibrium situations: $NE_1 = (0, 0)$, $NE_2 = (1, 1)$, $NE_3 = (2/3, 1/3)$.

From computer calculations with steps $h = 0.1$, $\Delta x = \Delta y = 0.01$, $\Delta p = \Delta q = 0.1$ we constructed approximate value functions $\underline{w}_{1,h}^n$, $\overline{w}_{1,h}^n$, $\underline{w}_{2,h}^n$, $\overline{w}_{2,h}^n$ guaranteeing strategies $u_{1,h}^m$, $\upsilon_{2,h}^m$ and punishment strategies $u_{2,h}^m$, $\upsilon_{1,h}^m$.

The strategies $u_{1,h}^m$, $\upsilon_{2,h}^m$, $u_{2,h}^m$, $\upsilon_{1,h}^m$ were constructed as follows. In the phase state square $I \times I$ of system (2.12) there is a switch-over curve which divides the square into two parts. In one, half the values of the controls are zero, and in the other unity. Figure 1 shows the switch-over curves SW_1 and SW_2 for guaranteeing strategies $u_{2,h}^m$, $\upsilon_{1,h}^m$. The characteristic feature of these curves in that SW_1 passes through the saddle point SP_1 and the curve SW_2 through the saddle point SP_2. Above the curve SW_1 the values of the control $u_{1,h}^m$ are equal to unity, and below they are zero. To the right of SW_2 the values of the control $\upsilon_{2,h}^m$ are equal to unity, and to the left they are equal to zero.

Figure 1 also shows the acceptable trajectory $TR_1 = (x^\varepsilon(\cdot), y^\varepsilon(\cdot))$ generated by the guaranteeing strategies $u_{1,h}^m$, $\upsilon_{2,h}^m$. Trajectory TR_1 leaves the initial position $IP = (1/5, 4/5)$ and tends to the Nash equilibrium situation NE_1. It consists of two-characteristics of the Hamilton–Jacobi equation (6.1) which are sections of straight lines pointing to the vertices of the square. Switching-over from one characteristic to the other occurs at the switch-over curve (which in this case is SW_1).

In the second case (one equilibrium) we considered the following almost-antagonistic payoff matrices A^1 and A^2

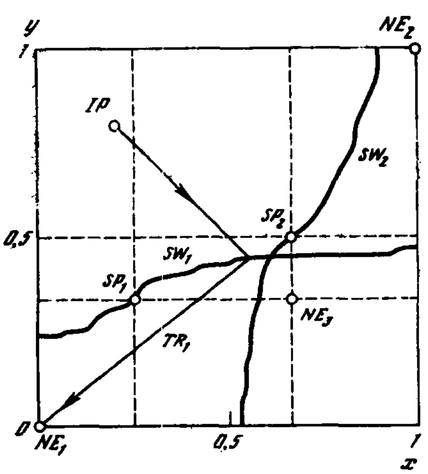

Fig. 1.

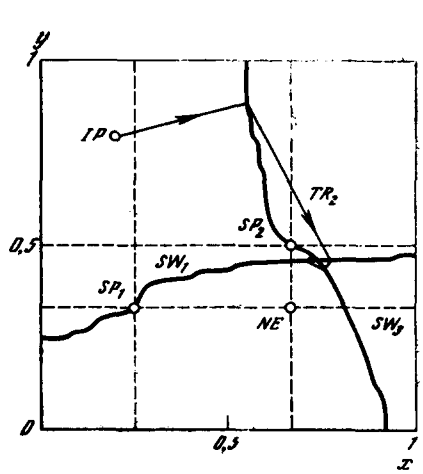

Fig. 2.

$$A^1 = B_1, \quad A^2 = B_3 = \begin{Vmatrix} 2 & 4 \\ 5 & 1 \end{Vmatrix}, \quad C_2 = a_{11}^2 - a_{12}^2 - a_{21}^2 + a_{22}^2 = -6$$

$$\alpha_1^2 = a_{22}^2 - a_{12}^2 = -3, \quad \alpha_2^2 = a_{22}^2 - a_{21}^2 = -4$$

In the corresponding antagonistic matrix games the saddle-point situations are the points $SP_1 = (1/4, 1/3)$ for matrix A^1 and $SP_2 = (2/3, 1/2)$ for A^2. In the non-antagonistic bimatrix game there is only one Nash equilibrium situation $NE = (2/3, 1/3)$.

The following numerical results were obtained for the evolutionary game of two coalitions with payoff matrices $A^1 = B_1, A^2 = B_3$. As in the first case, the guaranteeing strategy $u_{1,h}^m$ is given by the switch-over curve SW_1. The structure of the guaranteeing strategy $v_{2,h}^m$ is determined by the switch-over curve SW_3, which is shown in Fig. 2. This curve, like SW_2, passes through the saddle-point SP_2, but has a different direction. To the left of SW_3 the values of the control $u_{2,h}^m$ are unity, and to the right they are zero.

Figure 2 also shows the acceptable trajectory $TR_2 = (x^\varepsilon(\cdot), y^\varepsilon(\cdot))$ generated by the guaranteeing strategies $u_{1,h}^m$, $v_{2,h}^m$. Trajectory TR_2 leaves the initial position $IP = (1/5, 4/5)$ and approaches the point of intersection of the switch-over curves SW_1 and SW_2. It consists of segments of the characteristics of Eq. (6.1) directed towards the vertices of the square. Switch-over from one characteristic to another occurs at the switch-over curves SW_1, SW_3. In other words, trajectory TR_2 has "evolutionary–revolutionary" properties: "evolution" takes place in the characteristic intervals and "revolution" at the switch-over curves.

A feature of trajectory TR_2 is that it does not converge to the Nash equilibrium situation NE, which is typical of the trajectories in classical evolutionary models with replicatory dynamics [7–10], but tends to a new stable situation: the point of intersection of the switch-over curves SW_1 and SW_3. We note that the values of the global payoff functionals J_m of both coalitions on the acceptable trajectory TR_2 are better than their values on trajectories leaving the same initial position IP, but tending to the Nash equilibrium position NE.

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